If, instead of quitting the sphere when $R' = 0$, the rolling sphere is constrained to move on the surface of the fixed sphere until $\cos\varphi = 0$, then will the solution of (6) give the time $t$, in which the rolling sphere will descend from the summit of the fixed sphere over one fourth of its circumf., viz.;

$$t = \frac{1}{\sqrt{a}}\left[\frac{2}{\sqrt{2}}\log\frac{\sqrt{(1+\cos\varphi)} - \sqrt{2}}{\sqrt{(1-\cos\varphi)}}\right]_0^{\frac{\pi}{2}},$$

where $a = 10g \div 7(R+r)$.

This value of $t$, is not $[2 \div \sqrt{(2a)}]\log(1-\sqrt{2})$, as indicated at p. 122, $[2 \div \sqrt{(2a)}]\log(1-\sqrt{2})$ being the value of the integral at the superior limit only. At the inferior limit we have

$$\frac{\sqrt{(1+\cos\varphi)}-\sqrt{2}}{\sqrt{(1-\cos\varphi)}} = \frac{0}{0}, \text{ a vanishing fraction, whence}$$

$$\frac{-\sqrt{(1-\cos\varphi)}}{\sqrt{(1+\cos\varphi)}} = \frac{-0}{\sqrt{2}} = -0.$$

$$\therefore\ t = \frac{2}{\sqrt{(2a)}}\log\frac{1-\sqrt{2}}{-0} = \frac{2}{\sqrt{(2a)}}\log\frac{\sqrt{2}-1}{0} = \infty.$$

That is, the time required, to roll from the *summit* of the fixed sphere over one fourth of its circumference, is infinite.

If, instead of starting from the summit of the fixed sphere, the rolling sphere should start from a point, say $1''$ from the summit, the time, obviously, would be finite. In that case the time would be, as calculated by Mr. Kummell, $t = \sqrt{(2 \div a)} \times 12.7418368$.

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## *SOLUTIONS OF PROBLEMS IN NUMBER FIVE.*

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SOLUTIONS of problems in No. 5 have been received as follows:

From R. J. Adcock, 277; Alex. S. Christie, 276, 278, 279, 280; Prof. W. P. Casey, 278; G. M. Day, 276, 278; Prof. E. J. Edmunds, 276, 277, 278; Henry Gunder, 276, 278, 280; Henry Heaton, 276, 277, 278, 279, 280; Chas. H. Kummell, 276, 277, 278, 279, 280; Prof. D. J. McAdam, 279; O. L. Mathiot, 276; W. L. Marcy, 276, 278, 279; K. S. Putnam, 276; P. Richardson, 276; Prof. E. B. Seitz, 276, 278, 279, 280; Prof. J. Scheffer, 278, 280; Prof. D. Trowbridge, 278, 279, 280.

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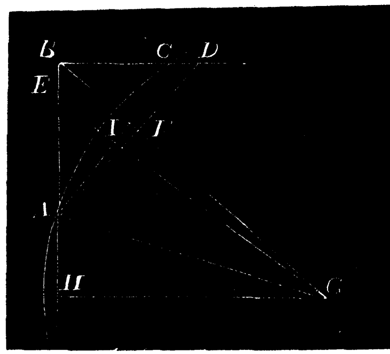
276. "Given the chord $AC$ of a circle, the side $AB$ of a right angled triangle constructed on $AC$ as hypothenuse, and the length of a perpendicular from $A$ upon the line joining the right angle at $B$ with the centre of the circle; to find the radius of the circle."

SOLUTION BY OCTAVIAN L. MATHIOT, BALTIMORE, MARYLAND.



Let $AC = a$, represent the given chord, $AB = b$, the given side of the right angled triangle $ABC$, and $AF = c$, the given perpendicular upon $BG$.

Bisect $AC$ in $I$ and through $I$ draw $EG$, at right angles with $AC$. Then, from the similar triangles $ABC$ and $AIE$, we find

$$AE = AC^2 \div 2AB = a^2 \div 2b. \quad (1)$$

$$\therefore \; EB = AB - AE = b - a^2 \div 2b. \quad (2)$$

From the similar triangles $BFA$ and $ABD$ we find

$$AD = AB^2 \div AF = b^2 \div c; \qquad (3)$$

and from the right angled triangle $ABD$ we have

$$BD = \sqrt{(AD^2 - AB^2)} = b\sqrt{(b^2-c^2)} \div c; \qquad (4)$$

$$\therefore \; CD = BD - BC = [b\sqrt{(b^2-c^2)} \div c] - \sqrt{(a^2-b^2)}. \qquad (5)$$

From the similar triangles $GEB$ and $ACD$ we find

$$BG = \frac{AD.EB}{CD} = \left[\frac{b^2}{c} \cdot \frac{2b^2 - a^2}{2b}\right] \div \left[\frac{b}{c}\sqrt{(b^2-c^2)} - \sqrt{(a^2-b^2)}\right]. \qquad (6)$$

And from the similar triangles $ABD$ and $GHB$ we find

$$BH = \frac{BG.BD}{AD} = \left[(b^2 - \tfrac{1}{2}a^2)\frac{1}{c}\sqrt{(b^2-c^2)}\right] \div \left[\frac{b}{c}\sqrt{(b^2-c^2)} - \sqrt{(a^2-b^2)}\right] \quad (7)$$

$$GH = \frac{BA.BG}{AD} = \left[b^2 - \tfrac{1}{2}a^2\right] \div \left[\frac{b}{c}\sqrt{(b^2-c^2)} - \sqrt{(a^2-b^2)}\right]$$

$$= \frac{b^2c - \frac{1}{2}a^2c}{b\sqrt{(b^2-c^2)} - c\sqrt{(a^2-b^2)}}. \qquad (8)$$

Also, we have

$$AH = BH - BA = \frac{bc\sqrt{(a^2-b^2)} - \frac{1}{2}a^2\sqrt{(b^2-c^2)}}{b\sqrt{(b^2-c^2)} - c\sqrt{(a^2-b^2)}}. \qquad (9)$$

Finally we have $GA = \sqrt{(GH^2 + AH^2)}$

$$= \sqrt{\left(\frac{[b^2c - \frac{1}{2}a^2c]^2 + [bc\sqrt{(a^2-b^2)} - \frac{1}{2}a^2\sqrt{(b^2-c^2)}]^2}{[b^2\sqrt{(b^2-c^2)} - c\sqrt{(a^2-b^2)}]^2}\right)}$$

$$= \frac{a\sqrt{a^2b^2 - 4bc\sqrt{[(a^2-b^2)(b^2-c^2)]}}}{2[b\sqrt{(b^2-c^2)} - c\sqrt{(a^2-b^2)}]}.$$

[Prof. Seitz and P. Richardson each obtains the same result as the above but by a somewhat different construction and calculation.]

277. "Solve, algebraically, the equation $x^{17}-1=0$."

SOLUTION BY R. J. ADCOCK, MONMOUTH, ILLINOIS.

All the roots of this equation are contained in the form

$$x=\cos\tfrac{1}{17}(2n\pi)\pm\sin\tfrac{1}{17}(2n\pi\sqrt{-1}),$$

where $n$ may be any entire number not greater than the half of 17, that is $n=0, 1, 2, 3, 4, 5, 6, 7, 8$, giving the seventeen roots,

$$\begin{aligned}
x&=1,\\
x&=\cos\tfrac{2}{17}\pi\pm\sin\tfrac{2}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{4}{17}\pi\pm\sin\tfrac{4}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{6}{17}\pi\pm\sin\tfrac{6}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{8}{17}\pi\pm\sin\tfrac{8}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{10}{17}\pi\pm\sin\tfrac{10}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{12}{17}\pi\pm\sin\tfrac{12}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{14}{17}\pi\pm\sin\tfrac{14}{17}\pi\sqrt{-1},\\
x&=\cos\tfrac{16}{17}\pi\pm\sin\tfrac{16}{17}\pi\sqrt{-1}.
\end{aligned}$$

Hence the binomial and quadratic factors of $x^{17}-1=0$, are

$$\begin{aligned}
x^{17}-1=0=(x-1)&(x^2-2x\cos\tfrac{2}{17}\pi+1)(x^2-2x\cos\tfrac{4}{17}\pi+1)\\
&(x^2-2x\cos\tfrac{6}{17}\pi+1)(x^2-2x\cos\tfrac{8}{17}\pi+1)(x^2-2x\cos\tfrac{10}{17}\pi+1)\\
&(x^2-2x\cos\tfrac{12}{17}\pi+1)(x^2-2x\cos\tfrac{14}{17}\pi+1)(x^2-2x\cos\tfrac{16}{17}\pi+1).
\end{aligned}$$

Similar expressions are given, Art. 40, p. 19, of Peirce's Curves, Functions and Forces, Vol. II. 1846.

SOLUTION BY HENRY HEATON, ATLANTIC, IOWA.

Factoring we have $(x-1)(x^{16}+x^{15}+x^{14}+\ldots+x+1)=0,$ (1)

$$\therefore x=1, \text{ or } x^{16}+x^{15}+x^{14}\ldots x=-1. \quad (2)$$

Since $x^{17}=1$, $x^{18}=x$, $x^{19}=x^2$, &c.; $\therefore$ we have

$$(x+x^2+x^4+x^8+x^{16}+x^{15}+x^{13}+x^9)(x^3+x^6+x^{12}+x^7+x^{14}+x^{11}+x^5+x^{10})$$
$$=4(x^{16}+x^{15}+x^{14}+\ldots+x)=-4; \quad (3)$$

$$\therefore x+x^2+x^4+x^8+x^{16}+x^{15}+x^{13}+x^9=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{17}, \quad (4)$$

$$x^3+x^6+x^{12}+x^7+x^{14}+x^{11}+x^5+x^{10}=-\tfrac{1}{2}\mp\tfrac{1}{2}\sqrt{17}, \quad (5)$$

$$(x+x^4+x^{16}+x^{13})(x^2+x^8+x^{15}+x^9)=-1. \quad (6)$$

Combining equations (4) and (6), we find

$$x+x^4+x^{16}+x^{13}=-\tfrac{1}{4}\pm\tfrac{1}{4}\sqrt{17}\pm\tfrac{1}{4}\sqrt{(34\mp2\sqrt{17})}, \quad (7)$$

$$x^2+x^8+x^{15}+x^9=-\tfrac{1}{4}\pm\tfrac{1}{4}\sqrt{17}\mp\tfrac{1}{4}\sqrt{(34\pm2\sqrt{17})}. \quad (8)$$

Similarly we find

$$x^3+x^{12}+x^{14}+x^5 = -\tfrac{1}{4}\mp\tfrac{1}{4}\sqrt{17}\pm\tfrac{1}{4}\sqrt{(34\pm2\sqrt{17})}, \qquad (9)$$

$$x^6+x^7+x^{11}+x^{10} = -\tfrac{1}{4}\mp\tfrac{1}{4}\sqrt{17}\mp\tfrac{1}{4}\sqrt{(34\pm2\sqrt{17})}, \qquad (10)$$

$$(x+x^{16})(x^4+x^{13}) = x^5+x^{14}+x^3+x^{12} = -\tfrac{1}{4}\mp\sqrt{17}\pm\tfrac{1}{4}\sqrt{(34\pm2\sqrt{17})}. \quad (11)$$

Combining (7) and (11) we find

$$x+x^{16} = \tfrac{1}{8}[-1\pm\sqrt{17}\pm\sqrt{(34\pm2\sqrt{17})}]\pm\tfrac{1}{4}[17\pm3\sqrt{17}\mp2\sqrt{(34\pm2\sqrt{17})}$$
$$\mp\sqrt{(34\mp2\sqrt{17})}]^{\frac{1}{2}}.$$

Since $x^{17} = 1$, $x^{16} = 1\div x$; $x = \tfrac{1}{16}[-1\pm\sqrt{17}\pm\sqrt{(34\mp2\sqrt{17})}]$
$\pm\tfrac{1}{8}[17\pm3\sqrt{17}\mp2\sqrt{(34\pm2\sqrt{17})}\mp\sqrt{(34\mp2\sqrt{17})}]^{\frac{1}{2}}\pm\tfrac{1}{8}\{-34\pm2\sqrt{17}$
$\mp2\sqrt{(34\mp2\sqrt{17})}\pm4[17\pm3\sqrt{17}\pm2\sqrt{(34\pm2\sqrt{17}}\pm\sqrt{(34\mp2\sqrt{17})}]^{\frac{1}{2}}\}^{\frac{1}{2}}$.

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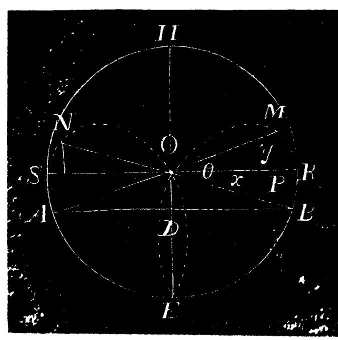
278. "A Chord $AB$ of a circle, whose centre is at $O$, remains constantly parallel to itself; $A$ and $B$ are connected to the centre $O$, and on $AO$ and $BO$, or those lines produced, $M$ and $N$ are taken so that $AM = BN = AB$. Required the locus of $M$ and $N$."

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $SR$ and $HE$ represent two diameters at right angles to each other, $SR$ being parallel to $AB$. Let $\angle POM = \theta$; $PO = x$ and $PM = y$. Then $OM = \sqrt{(x^2+y^2)}$, $AO\times\cos\theta = AD$, or $2r\cos\theta = 2AD = AM = r+\sqrt{(x^2+y^2)}$; therefore $(2\cos\theta-1)r = \sqrt{(x^2+y^2)}$, or $(2\cos\theta-1)^2r^2 = x^2+y^2$, which is the equation of the locus of $M$. By giving in succession values to $\theta$, and taking the corresponding values of $AM$, the curve which is the locus of $M$ will be traced. The curve will pass through the points $E$, $O$ and $R$. The locus of $N$ will be a similar curve and will pass through the points $E$, $O$ and $S$.




SOLUTION BY D. TROWBRIDGE, A. M., WATERBURGH, N. Y.

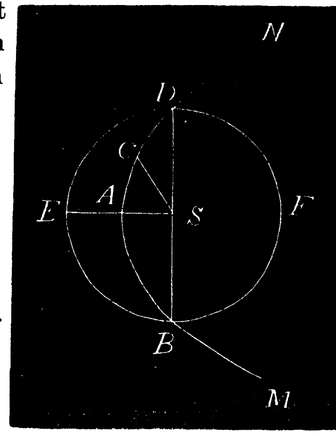
Put the radius of the circle$=a$, co-ordinates of $M=x$, $y$, and of $A$ $x'$ $y'$. We now have $AB = AO\pm OM$, or $2x' = a\pm\sqrt{(x^2+y^2)}$, and $xy' = x'y = x\sqrt{(a^2-x'^2)}$, $x'^2(x^2+y^2) = a^2x^2$, and $2x' = 2ax\div\sqrt{(x^2+y^2)} = a\pm\sqrt{(x^2+y^2)}$. Put $x^2+y^2 = r^2$, and $x = r\cos\theta$, then $2a\cos\theta = a\pm r$,

$$r = a(2\cos\theta-1), \qquad r = a(1-2\cos\theta).$$

These are the eq'ns of the trisectrix. (See Hadden's Diff. Cal., p. 110.)

279. "If the earth be projected, when in perihelion, at right angles to a line joining the Earth and Sun, with a velocity sufficient to cause it to describe a parabola, how long would it take it to reach the orbit of Jupiter? The orbits of the Earth and Jupiter supposed in the same plane, both circles, and the radius of Jupiter's orbit five times that of the Earth."

SOLUTION BY PROF. D. J. MC ADAM, WASHINGTON, PA.




As in problem 50, ANALYST, Vol. II, let the circle $DEBF$ be the orbit of Jupiter, then the area $ASD$ ($D$ being the point of exit from Jupiter's orbit)

$$= R^2(\tan\tfrac{1}{2}\theta + \tfrac{1}{3}\tan^3\tfrac{1}{2}\theta) = \tfrac{14}{3}R^2.$$

(Where $\theta = \angle ASC$.)

Then, putting $\tan\frac{1}{2}\theta = 2$ (see ANALYST, p. 30, Vol. II),

$$\sqrt{(10R)} : \sqrt{(4R)} :: 25\pi R^2 : 50\pi R^2$$

= the area described by the earth in one of Jupiter's years.

Therefore the time it will take the Earth to describe the portion of its orbit within Jupiter's orbit will $= \frac{14}{3}R^2 \div 50\pi R^2$, of Jupiter's year, $= 389.3$ days.

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280. "Find the sum of the series

$$x\sin a - \frac{x^2\sin 2a}{1.2} + \frac{x^3\sin 3a}{1.2.3} - \frac{x^4\sin 4a}{1.2.3.4} + \ldots \text{ to infinity,}$$

$$1 - x\cos a + \frac{x^2\cos 2a}{1.2} - \frac{x^3\cos 3a}{1.2.3} + \frac{x^4\cos 4a}{1.2.3.4} - \ldots \text{ to infinity."}$$

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURV., DETROIT, MICH.

We have $s_2 \pm s_1 i = 1 - xe^{\mp ai} + \dfrac{x^2}{1.2}e^{\mp 2ai} - \dfrac{x^3}{1.2.3}e^{\mp 3ai} + \ldots$

$$= e^{-xe^{\mp ai}} \text{ (where } i = \sqrt{-1}), \qquad (1)$$

then $$\sqrt{(s_2^2 + s_1^2)} = e^{-x\cos a}, \qquad (2)$$

and $$\sqrt{\frac{s_2 + s_1 i}{s_2 - s_1 i}} = e^{+x\sin a}, \qquad (3)$$

or, taking logarithms,

$$xi\sin a = i\left[\frac{s_1}{s_2} - \frac{1}{3}\left(\frac{s_1}{s_2}\right)^3 + \frac{1}{5}\left(\frac{s_1}{s_2}\right)^5 - \ldots\right]$$

$$= i\tan^{-1}(s_1 \div s_2).$$

$$\therefore s_1 = s_2 \tan (x \sin \alpha). \tag{4}$$

Substituting this in (2) we obtain

$$s_2 = e^{-x \cos \alpha} \cos (x \sin \alpha),$$

and by (4)

$$s_1 = e^{-x \cos \alpha} \sin (x \sin \alpha).$$

[Mr. Kummell has also sent us a solution of prob. 266, done by a method similar to the above.]

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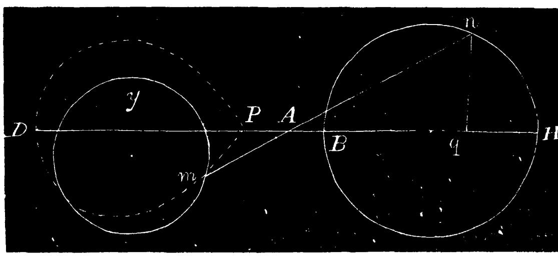
SOLUTION OF PROB. 85 (SEE P. 193, VOL. II) BY PROF. W. P. CASEY.

Let $mn$ be the given line, passing through the given point $A$, $BnH$ and $my$ the given circles.




Through the given p't $A$, draw the diameter $BH$, and make $HP$ and $BD$ each $= mn$, $\therefore P$ and $D$ are given points, and are in the locus of the point $m$.

Draw $ng$ perpendicular to $BH$. Let $Ag = x$, $gn = y$, $BH = d$, $AB = a$, $mn = h$; then $ng^2 = Bg \times gH$, or $y^2 = (x - a)(a + d - x) = (x - a)(c - x)$ (if $c = a + d$), and $y^2 = -x^2 + (a + c)x - ac$. Let $An = r$, $\angle A = \theta$, then $y = r \sin \theta$, $x = r \cos \theta$; $\therefore$ by substitution $r^2 \sin^2 \theta = -r^2 \cos^2 \theta + (a + c) r \cos \theta - ac$, and $r^2 - (a + c) r \cos \theta = -ac$;

$$\therefore r = \tfrac{1}{2} \left\{ (a + c) \cos \theta \pm \sqrt{[(a + c) \cos^2 \theta - 4ac]} \right\}.$$

Because $Am = mn - An = h - r$, by giving successive values to $\theta$, and taking the corresponding values of $r$, the curve, which is the locus of $m$, will be traced. The curve is an oval, whose axis $PD = BH = d$. The point $m$ is therefore given and the line $mn$ is in position.

[Prof. Casey also discusses the case when $An - Am$ is given, but our space will not permit its insertion.]

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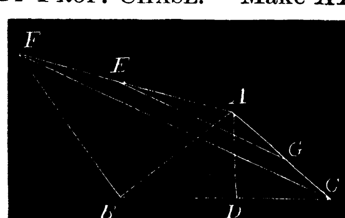
CONSTRUCTION OF THE METIAN RATIO BY PROF. CHASE.— Make $AB = 7$; $BC = 8$; $BD = 9$; $DF = 15$; $AE = AC$; $AB$ perpendicular to $CD$ and $DF$ perpendicular to $AD$, and draw $EG$ parallel to $FC$. $AF \div AG = 355 \div 113 = 3.14159292+$, $\pi$ being $3.14159265$.




The error is less than one one hundred and sixteen thousandths of one per cent.